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A SIZE-WIDTH INEQUALITY FOR DISTRIBUTIVE LATTICES*

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We show that every collection of w sets such that none contains any other generates at least $3w-2$ sets under the operations of taking intersections and unions. In particular, we prove that if the finite distributive lattice $\mathscr{L}$ contains an antichain of size w, then $|\mathscr{L}|\geqq 3w$, for $w\neq 1, 2, 3, 6$, where the minimal exceptional cases arise from the Boolean algebras $\mathscr{B}_n$ with $n=0, 1, 2, 3, 4$ atoms.

The main result

We assume the reader to be familiar with the basic notions and properties of (partially) ordered sets and, in particular, (distributive) lattices. Birkhoff [1] or any other standard textbook on lattice theory may serve as a reference for the terminology we use. Recall that the *width* of a finite lattice $\mathscr{L}$ is the size of a maximal cardinality antichain in $\mathscr{L}$.

The example of a chain shows that generally the size $|\mathscr{L}|$ of the lattice $\mathscr{L}$ cannot be bounded from above in terms of its width w. A trivial lower bound is given by $|\mathscr{L}|\geqq w+2$ for $w\geqq 2$. This lower bound is attained infinitely often by the class of modular lattices M_n of length 2 with $n\geqq 2$ atoms. For the class of finite distributive lattices, however, a nontrivial lower bound exists. In particular, we show

Theorem. *Let $\mathscr{L}$ be a finite distributive lattice containing an antichain of size w. Then*

a) *If $w\neq 1, 2, 3, 6$, then $|\mathscr{L}|\geqq 3w$.*

b) *If $w=1, 2, 3, 6$, then $|\mathscr{L}|\geqq 3w-2$. Moreover, $|\mathscr{L}|=3w-1$ iff $\mathscr{L}=\mathscr{B}_n$ $(n=1, 3)$ and $|\mathscr{L}|=3w-2$ iff $\mathscr{L}=\mathscr{B}_n$ $(n=0, 2, 4)$, where $\mathscr{B}_n$ denotes the Boolean algebra with n atoms.*

Corollary. *Given a family W of w arbitrary sets such that none contains any other, at least $3w-2$ sets are obtained by forming intersections and unions.*

Proof. W is an antichain in the finite distributive lattice $\mathscr{L}(W)$ generated by W with respect to intersections and unions. ∎

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We prove the Theorem by induction on $|\mathscr{L}|$, noting that the statement trivially is true for $|\mathscr{L}|=1$. Assuming the statement to be true for all distributive lattices $\mathscr{L}'$ of size $|\mathscr{L}'|<|\mathscr{L}|$, we verify the Theorem for $\mathscr{L}$ by induction on w. We devote the next section to the latter induction.

It will be convenient to think of $\mathscr{L}$ as being represented as the collection of all order ideals of its ordered set P of join-irreducible elements.

Fix a maximal element $e\in P$. Then $\mathscr{L}$ is partitioned into the non-empty intervals

$$\mathscr{L}^o = \{A\in\mathscr{L}\colon\ e\notin A\} \quad \text{and}$$

$$\mathscr{L}^e = \{A\in\mathscr{L}\colon\ e\in A\}.$$

Note that, for every $\mathscr{K}\subseteqq\mathscr{L}^e$, $\mathscr{K}/e=\{A-e\colon A\in\mathscr{K}\}$ is an ordered subset of $\mathscr{L}^o$. In particular, $\mathscr{L}^e/e$ is an upper interval of $\mathscr{L}^o$, isomorphic to $\mathscr{L}^e$. Hence $|\mathscr{L}^o|\geqq|\mathscr{L}^e|$. Moreover, if $\mathscr{L}^o$ is a Boolean algebra, every interval of $\mathscr{L}^o$ has cardinality 2^k for some $k\in\mathbf{N}$ and therefore $|\mathscr{L}^e|=2^k$ for some $k\in\mathbf{N}$. We will repeatedly make use of this observation. Also note that if $\mathscr{L}^o\simeq\mathscr{B}_n\simeq\mathscr{L}^e$ for some $n\in\mathbf{N}$, then necessarily $\mathscr{L}\simeq\simeq\mathscr{B}_{n+1}$.

A word to the notation and terminology: a k-element (anti)chain will simply be called a k-(anti)chain. Furthermore, if W is a w-antichain in $\mathscr{L}$, we let $W^o=W\cap\mathscr{L}^o$ and $W^e=w\cap\mathscr{L}^e$ and $w^o=|W^o|$, $w^e=|W^e|$ and hence $w=w^o+w^e$.

Proof of the Theorem

The main induction is carried out in a sequence of lemmas. The method of proof for each of those lemmas is as follows: suppose that the statement fails and that the antichain W in question provides a counterexample of minimal size w. Decomposition of $\mathscr{L}$ into $\mathscr{L}^o$ and $\mathscr{L}^e$ and the examination of several cases which can occur will then yield a contradiction based on the induction hypothesis with respect to $|\mathscr{L}|$, i.e., on the assumption that each lemma is true for $\mathscr{L}^o$ and $\mathscr{L}^e$. Thus the case $w^e=0$ need not be considered.

Some preliminary observations are useful in order to reduce the amount of case checking.

A) If every member of W^e is a coatom of $\mathscr{L}^e$, then $W'=W^o\cup W^e/e$ is an anti-chain in $\mathscr{L}^o$ of size $w'=w$ unless $W^o=\{P-e\}$. But this latter exceptional case is trivial.

B) If $w^e=2$, we will assume $|\mathscr{L}^e|\geqq5$ since $|\mathscr{L}^e|=4$ implies that W^e is the set of coatoms of $\mathscr{L}^e$.

C) If $w^e=2$ and $|\mathscr{L}^e|=5$, then the interval $\mathscr{L}^e/e$ can contain at most one member of W^o. Indeed, if $A, B\in W^o$ are such that $A, B\in\mathscr{L}^e/e$ and $A\neq B$, then necessarily $\{A, B\}=W^e/e$ and hence W is not an antichain. So $\mathscr{L}^o$ contains the antichain $W'=W^e/e\cup(W^o-\mathscr{L}^e/e)$ of size $w'=w^o+1$. We will use this principle later also in a slightly generalized form.

D) $\mathscr{L}^o$ always contains a w^e-antichain.

Those four principles A, B, C, and D are essential to our case analysis. Principle D says in particular that it suffices to consider the cases where $w^o \geqq w^e$. We now proceed with the proof of the Theorem.

Lemma 1. *If $W=\{A, B, C\}$ is a 3-antichain of $\mathscr{L}$, then $|\mathscr{L}|\geqq 8$. Moreover, if $|\mathscr{L}|=8$, then $\mathscr{L}\cong\mathscr{B}_3$.*

Proof. If $w^e=0$, then $|\mathscr{L}|=|\mathscr{L}^o|+|\mathscr{L}^e|\geqq 8+|\mathscr{L}^e|$ by induction on $|\mathscr{L}|$. If $w^e=2$, then $|\mathscr{L}^e|\geqq 4$ and thus $|\mathscr{L}|\geqq 4+4=8$. Furthermore, $|\mathscr{L}|=8$ holds if and only if $|\mathscr{L}^o|=|\mathscr{L}^e|=4$, i.e., $\mathscr{L}^o\cong\mathscr{B}_2\cong\mathscr{L}^e$ and thus $\mathscr{L}\cong\mathscr{B}_3$.

If $w^e=1$ and $\mathscr{L}^e$ is not a chain, then $|\mathscr{L}^e|\geqq 4$ and hence $|\mathscr{L}|\geqq 8$. If $\mathscr{L}^e$ is a chain, then $\mathscr{L}^o$ contains a 3-antichain in the case where $\mathscr{L}^e$ is a 2-chain, by principle A, and hence $|\mathscr{L}|\geqq 9$. If $\mathscr{L}^e$ is at least a 3-chain, then $\mathscr{L}^o$ must have an upper interval isomorphic to a 3-chain and therefore $|\mathscr{L}^o|\geqq 6$, i.e., $|\mathscr{L}|\geqq 10$. Thus again $|\mathscr{L}|=8$ is only possible for $\mathscr{L}^o\cong\mathscr{B}_2\cong\mathscr{L}^e$ and $\mathscr{L}\cong\mathscr{B}_3$. ∎

Lemma 2. *If $W\subseteqq\mathscr{L}$ is a 4-antichain, then $|\mathscr{L}|\geqq 12$.*

Proof. If $w^e=1$ and $\mathscr{L}^e$ is a 2-chain, then $|\mathscr{L}^o|\geqq 12$ by induction on $|\mathscr{L}|$ and principle A above. If $\mathscr{L}^e$ is a 3-chain, $\mathscr{L}^o\cong\mathscr{B}_3$ is impossible since no interval of $\mathscr{B}_3$ is isomorphic to a 3-chain. So $|\mathscr{L}^o|\geqq 9$ by Lemma 1. Otherwise $|\mathscr{L}^e|\geqq 4$ and hence $|\mathscr{L}|\geqq 12$.

If $w^e=2$, we may assume $|\mathscr{L}^e|=5$, by principle B. By principle C, $\mathscr{L}^o$ contains a 3-antichain and hence $|\mathscr{L}|=|\mathscr{L}^o|+|\mathscr{L}^e|\geqq 8+5=13$.

If $w^e=3$, $|\mathscr{L}|\geqq 16$ by principle D and Lemma 1. ∎

Lemma 3. *If $W\subseteqq\mathscr{L}$ is a 5-antichain, then $|\mathscr{L}|\geqq 15$.*

Proof. If $w^e=1$, we may assume that $\mathscr{L}^e$ contains a 3-chain or a 2-antichain, i.e., $|\mathscr{L}^e|\geqq 15$ by Lemma 2.

If $w^e=2$ and $|\mathscr{L}^e|\geqq 7$, then $|\mathscr{L}|\geqq 8+7=15$. If $|\mathscr{L}^e|<7$, we may assume that $\mathscr{L}^o$ contains a 4-antichain, by principles B and C, and thus $|\mathscr{L}|\geqq 12+5=17$ since $|\mathscr{L}^e|=6$ implies $\mathscr{L}^o\not\cong\mathscr{B}_3$ and hence $|\mathscr{L}|\nless 15$. ∎

Lemma 4. *If $W\subseteqq\mathscr{L}$ is a 6-antichain, then one of the following mutually exclusive cases occurs:*

a) $|\mathscr{L}|=16$ *and* $\mathscr{L}\cong\mathscr{B}_4$.

b) $|\mathscr{L}|=17$ *and* $\mathscr{L}-P\cong\mathscr{B}_4$ *or* $\mathscr{L}-\emptyset\cong\mathscr{B}_4$.

c) $|\mathscr{L}|\geqq 18$.

Proof. Assume that neither P is join-irreducible nor $\emptyset$ is meet-irreducible in $\mathscr{L}$. We show that either c) or a) occurs.

If $w^e=1$, we may assume $|\mathscr{L}^e|\geqq 3$, by principle A, and thus $|\mathscr{L}|\geqq 18$ by Lemma 3.

If $w^e=2$ and W^e is a set of coatoms of $\mathscr{L}^e$, we have $|\mathscr{L}|=|\mathscr{L}^o|+|\mathscr{L}^e|\geqq 16+4=20$, by principle A. If W^e is a set of atoms of $\mathscr{L}^e$ and $|\mathscr{L}^e|=5$, we conclude $|\mathscr{L}|=|\mathscr{L}^o|+|\mathscr{L}^e|\geqq 15+5=20$ from principle C. If $|\mathscr{L}^e|\geqq 6$, clearly $|\mathscr{L}|\geqq 18$ by Lemma 2.

If $w^e=3$ and $|\mathscr{L}^e|\geqq 9$, then $|\mathscr{L}|\geqq 18$. If $|\mathscr{L}^e|=8$, i.e., $\mathscr{L}^e\cong\mathscr{B}_3$, then $\mathscr{B}_3$ occurs as an upper interval of $\mathscr{L}^o$. Hence $|\mathscr{L}^o|=8$ implies $\mathscr{L}^o\cong\mathscr{B}_3\cong\mathscr{L}^e$ and thus

$\mathcal{L} \cong \mathcal{B}_4$. If $|\mathcal{L}^o| = 9$, $\emptyset$ must be meet-irreducible in $\mathcal{L}^o$ and hence in $\mathcal{L}$. Finally, $|\mathcal{L}^o| \geqq 10$ yields $|\mathcal{L}| \geqq 18$. ▮

We are now ready to deal with the general case.

Lemma 5. *If $\mathcal{L}$ contains an antichain W of size $w \geqq 7$, then $|\mathcal{L}| \geqq 3w$.*

Proof. Suppose that the lemma fails and that W provides a counterexample of minimal cardinality. Examining six cases, we will derive a contradiction.

(i) $w^e = 1$. By principle A, $\mathcal{L}^e$ must contain a 3-chain. If $w^o \geqq 7$, $|\mathcal{L}| \geqq \geqq 3w^o + 3 = 3w$ by induction on $|\mathcal{L}|$. So assume $w^o = 6$. If $\mathcal{L}^e$ is a 3-chain and $|\mathcal{L}^o| < 18$, then necessarily $\mathcal{L}^o - (P-e) \cong \mathcal{B}_4$ (Lemma 4) since $\mathcal{B}_4$ contains no upper interval isomorphic to a 3-chain. Furthermore, e must be above at least 3 atoms of $\mathcal{L}^o$ because otherwise $\mathcal{L}^e$ has at least 2 atoms and cannot be a 3-chain. But this is impossible since W^o must consist of all 2-element subsets of the 4-atoms of $\mathcal{L}^o$ and hence e is comparable with 3 members of W^o, i.e., W is not an antichain. Similarly, $\mathcal{L}^e$ cannot be a 4-chain. So $|\mathcal{L}^e| \geqq 5$ and $|\mathcal{L}| \geqq 21$.

(ii) $w^e = 2$. If $|\mathcal{L}^e| = 5$, $\mathcal{L}^o$ contains an antichain of size $w' = w^o + 1$ (Principle C). If $w' \geqq 7$, we have $|\mathcal{L}| \geqq 3w$ by induction and, if $w' \geqq 6$, by the preceding lemmas. If $|\mathcal{L}^e| = 6$ or 7, the lemma can only fail when $w^o = 6$. Neither case a) nor b) of Lemma 4 allows an upper interval of $\mathcal{L}^o$ with 6 or 7 elements. Hence $|\mathcal{L}^o| \geqq 18$ and $|\mathcal{L}| \geqq 3w$. If $|\mathcal{L}^e| \geqq 8$, $|\mathcal{L}| \geqq 3w$ follows immediately.

(iii) $w^e = 3$. If $|\mathcal{L}^e| \geqq 9$, the lemma could only fail in case b) of Lemma 4 for $\mathcal{L}^o$ and $\mathcal{L}^e - P \cong \mathcal{B}_3$ (in case a), $\mathcal{L}^e$ would have at least 16 elements). So e must be above at least one atom of $\mathcal{L}^o$. Now W^o consists of all 2-element subsets of atoms of $\mathcal{L}^o$. Hence every member of W^e contains a member of W^o as a subset and W is not an antichain. Therefore $|\mathcal{L}^e| = 8$ and $\mathcal{L}^e \cong \mathcal{B}_3$. By principle A, W^e is the set of atoms of $\mathcal{L}^e$. If the interval $\mathcal{L}^e/e$ of $\mathcal{L}^o$ contains no member of W^o, then we conclude that $W' = W^o \cup W^e/e$ is an antichain of size $w' = w$ in $\mathcal{L}^o$ as in principle. C. If there exists $A \in W^o$ with $A \in \mathcal{L}^e/e$, A necessarily is a coatom of $\mathcal{L}^o$, i.e., there exists a maximal join-irreducible element a in $\mathcal{L}^o$ so that $P - e = A \cup a$. Decomposing $\mathcal{L}^o$ into $\mathcal{L}^{oo}$ and $\mathcal{L}^{oa}$, we see that $W^o \cap \mathcal{L}^{oo} = \{A\}$ and $|W^o \cap \mathcal{L}^{oa}| = w^o - 1$. Thus $|\mathcal{L}^o| \geqq |\mathcal{L}^e| + +3(w^o - 1) - 2$ and therefore $|\mathcal{L}| = |\mathcal{L}^o| + |\mathcal{L}^e| \geqq 3w^o + 11 \geqq 3w$.

(iv) $w^e = 4$. The lemma can only fail if case a) or b) of Lemma 4 occurs for $\mathcal{L}^o$. Since $\mathcal{L}^e$ is isomorphic to an upper interval of $\mathcal{L}^o$, $|\mathcal{L}^e| \geqq 16$ and $|\mathcal{L}| \geqq 32$.

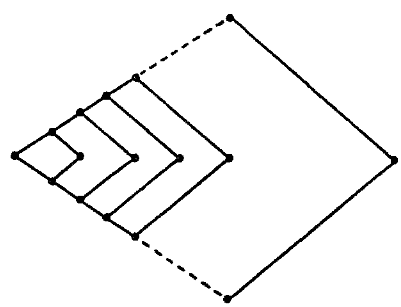
(v) $w^e = 5$. As in (iv), we conclude $|\mathcal{L}^e| \geqq 17$ if case b) of Lemma 4 occurs for $\mathcal{L}^o$. If a) occurs, $\mathcal{L}^e$ must be a Boolean algebra and hence $\mathcal{L}^e \cong \mathcal{B}_4$ implies $|\mathcal{L}| \geqq 32$.

(vi) $w^e = 6$. If $|\mathcal{L}^e| \geqq 18$, clearly $|\mathcal{L}| \geqq 3w$. So $|\mathcal{L}^e| \leqq 17$.

Now W^e consists of all 6 unions of pairs of atoms of $\mathcal{L}^e$ (the case where e is meet-irreducible in $\mathcal{L}^e$ can be treated the same way). Hence the interval $\mathcal{L}^e/e$ of $\mathcal{L}^o$ can contain at most 4 members of W^o. Thus $\mathcal{L}^o$ contains the antichain $W' = = W^e/e \cup (W^o - (W^o \cap \mathcal{L}^e/e))$ of size $w' \geqq w_0 + 2$. Hence $|\mathcal{L}| \geqq 3w$, and the proof of both the lemma and the Theorem is complete since apparently the case "$w^e \geqq 7$" need not be considered. ▮

Remarks

The conjecture $|\mathcal{L}|\geqq 3w-2$ for the width w of the finite semi-distributive lattice $\mathcal{L}$, i.e. $\mathcal{L}$ such that $a\wedge b=a\wedge c$ implies $a\vee(b\wedge c))=a\wedge b$ and $a\vee b=a\vee c$ implies $a\vee(b\wedge c)=a\vee b$ for all $a, b, c\in\mathcal{L}$, is due to the second author. In the class of semi-distributive lattices, this bound is thight infinitely often. This can be seen by constructing semi-distributive lattices $\mathcal{L}_1, \mathcal{L}_2, \ldots$ inductively as follows: if the bound is thight for $\mathcal{L}_i$, let $\mathcal{L}_{i+1}$ be the disjoint union of $\mathcal{L}_i$ with any semi-distributive lattice for which the bound is tight together with a new zero and one:



Since only little is known about the general structure of semi-distributive lattices, it appears difficult to verify the conjecture in general.

Our Theorem yields a sharper lower bound for the subclass of distributive lattices. One feels, however, that this bound can possibly be improved in the sense that every linear lower bound admits only a finite number of exceptions in the class of distributive lattices. In fact, comparison of size and width of Boolean algebras suggests a lower bounds of order $w\sqrt{\log w}$. The proof of such a lower bound probably would call for different techniques than those used in the proof of Section 2.

A closely related open question concerns the role of Boolean algebras in the class of distributive lattices: is $\mathcal{B}_n$ the unique distributive lattice of minimal size allowing width $\binom{n}{[n/2]}$? Lemma 1 and Lemma 4 give an affirmative answer for $n\leqq 4$ and it is not difficult to settle the case $n=5$.

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Reference

[1] G. Birkhoff, *Lattice Theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ. 25, Providence, R. I., 1967.

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